

PARAPRODUCTS AND PRODUCTS OF FUNCTIONS IN $BMO(\mathbb{R}^n)$ AND $H^1(\mathbb{R}^n)$ THROUGH WAVELETS

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ABSTRACT. In this paper, we prove that the product (in the distribution sense) of two functions, which are respectively in $BMO(\mathbb{R}^n)$ and $\mathcal{H}^1(\mathbb{R}^n)$, may be written as the sum of two continuous bilinear operators, one from $\mathcal{H}^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$, the other one from $\mathcal{H}^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into a new kind of Hardy-Orlicz space denoted by $\mathcal{H}^{\log}(\mathbb{R}^n)$. More precisely, the space $\mathcal{H}^{\log}(\mathbb{R}^n)$ is the set of distributions f whose grand maximal function $\mathcal{M}f$ satisfies

$$\int_{\mathbb{R}^n} \frac{|\mathcal{M}f(x)|}{\log(e + |x|) + \log(e + |\mathcal{M}f(x)|)} dx < \infty.$$

The two bilinear operators can be defined in terms of paraproducts. As a consequence, we find an endpoint estimate involving the space $\mathcal{H}^{\log}(\mathbb{R}^n)$ for the div-curl lemma.

1. INTRODUCTION

Products of functions in \mathcal{H}^1 and BMO have been considered by Bonami, Iwaniec, Jones and Zinsmeister in [2]. Such products make sense as distributions, and can be written as the sum of an integrable function and a function in a weighted Hardy-Orlicz space. To be more precise, for $f \in \mathcal{H}^1(\mathbb{R}^n)$ and $g \in BMO(\mathbb{R}^n)$, we define the product (in the distribution sense) fg as the distribution whose action on the Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is given by

$$(1.1) \quad \langle fg, \varphi \rangle := \langle \varphi g, f \rangle,$$

where the second bracket stands for the duality bracket between $\mathcal{H}^1(\mathbb{R}^n)$ and its dual $BMO(\mathbb{R}^n)$. It is then proven in [2] that

$$(1.2) \quad fg \in L^1(\mathbb{R}^n) + \mathcal{H}_{\omega}^{\Phi}(\mathbb{R}^n).$$

Here $\mathcal{H}_{\omega}^{\Phi}(\mathbb{R}^n)$ is the weighted Hardy-Orlicz space related to the Orlicz function

$$(1.3) \quad \Phi(t) := \frac{t}{\log(e + t)}$$

and with weight $\omega(x) := (\log(e + |x|))^{-1}$.

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Our aim is to improve this result in many directions. The first one consists in proving that the space $\mathcal{H}_\omega^\Phi(\mathbb{R}^n)$ can be replaced by a smaller space. More precisely, we define the Musielak-Orlicz space $L^{\log}(\mathbb{R}^n)$ as the space of measurable functions f such that

$$\int_{\mathbb{R}^n} \frac{|f(x)|}{\log(e + |x|) + \log(e + |f(x)|)} dx < \infty.$$

The space $\mathcal{H}^{\log}(\mathbb{R}^n)$ is then defined, as usual, as the space of tempered distributions for which the grand maximal function is in $L^{\log}(\mathbb{R}^n)$. This is a particular case of a Hardy space of Musielak-Orlicz type, with a variable (in x) Orlicz function that is also called a Musielak-Orlicz function (see [13]). This kind of space had not yet been considered. A systematic study of Hardy spaces of Musielak-Orlicz type has been done separately by the last author [13]. It generalizes the work of Janson [12] on Hardy-Orlicz spaces. In particular, it is proven there that the dual of the space $\mathcal{H}^{\log}(\mathbb{R}^n)$ is the generalized BMO space that has been introduced by Nakai and Yabuta (see [19]) to characterize multipliers of $BMO(\mathbb{R}^n)$. Remark that by duality with our result, functions f that are bounded and in the dual of $\mathcal{H}^{\log}(\mathbb{R}^n)$ are multipliers of $BMO(\mathbb{R}^n)$. By the theorem of Nakai and Yabuta there are no other multipliers, which, in some sense, indicates that $\mathcal{H}^{\log}(\mathbb{R}^n)$ could not be replaced by a smaller space.

Secondly we answer a question of [2] by proving that there exists continuous bilinear operators that allow to split the product into an $L^1(\mathbb{R}^n)$ part and a part in this Hardy Orlicz space $\mathcal{H}^{\log}(\mathbb{R}^n)$. More precisely we have the following.

Theorem 1.1. *There exists two continuous bilinear operators on the product space $\mathcal{H}^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$, respectively $S : \mathcal{H}^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \mapsto L^1(\mathbb{R}^n)$ and $T : \mathcal{H}^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n) \mapsto \mathcal{H}^{\log}(\mathbb{R}^n)$ such that*

$$(1.4) \quad fg = S(f, g) + T(f, g).$$

The operators S and T are defined in terms of a wavelet decomposition. The operator T is defined in terms of paraproducts. There is no uniqueness, of course. In fact, the same decomposition of the product fg has already been considered by Dobyinsky and Meyer (see [9, 7, 8], and also [4, 5]). The action of replacing the product by the operator T was called by them a *renormalization* of the product. Namely, T preserves the cancellation properties of the factor, while S does not. Dobyinsky and Meyer considered L^2 data for both factors, and showed that $T(f, g)$ is in the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. What is surprising in our context is that both terms inherit some properties of the factors. Even if the product fg is not integrable, the function $S(f, g)$ is, while $T(f, g)$ inherits cancellation properties of functions in Hardy spaces without being integrable. So, in some way each term has more properties than expected at first glance.

Another implicit conjecture of [2] concerns bilinear operators with cancellations, such as the ones involved in the div-curl lemma for instance. In this case

it is expected that there is no L^1 term. To illustrate this phenomenon, it has been proven in [1] that, whenever F and G are two vector fields respectively in $\mathcal{H}^1(\mathbb{R}^n, \mathbb{R}^n)$ and $BMO(\mathbb{R}^n, \mathbb{R}^n)$ such that F is curl-free and G is div-free, then their scalar product $F \cdot G$ is in $\mathcal{H}_w^\Phi(\mathbb{R}^n, \mathbb{R}^n)$ (in fact there is additional assumption on the BMO factor). By using the same technique as Dobyinsky to deal with the terms coming from S , we give a new proof, without any additional assumption. Namely, we have the following.

Theorem 1.2. *Let F and G be two vector fields, one of them in $\mathcal{H}^1(\mathbb{R}^n, \mathbb{R}^n)$ and the other one in $BMO(\mathbb{R}^n, \mathbb{R}^n)$, such that $\text{curl } F = 0$ and $\text{div } G = 0$. Then their scalar product $F \cdot G$ (in the distribution sense) is in $\mathcal{H}^{\log}(\mathbb{R}^n)$.*

In Section 2 we introduce the spaces $L^{\log}(\mathbb{R}^n)$ and $\mathcal{H}^{\log}(\mathbb{R}^n)$ and give the generalized Hölder inequality that plays a central role when dealing with products of functions respectively in $L^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$. In Sections 3 and 4 we give prerequisites on wavelets and recall the L^2 estimates of Dobyinsky. We prove Theorem 1.1 in Section 5 and Theorem 1.2 in Section 6.

2. THE SPACE $\mathcal{H}^{\log}(\mathbb{R}^n)$ AND A GENERALIZED HÖLDER INEQUALITY

We first define the (variable) Orlicz function

$$\theta(x, t) := \frac{t}{\log(e + |x|) + \log(e + t)}$$

for $x \in \mathbb{R}^n$ and $t > 0$. For fixed x it is an increasing function while $t \mapsto \theta(x, t)/t$ decreases. We have $p < 1$ in the following inequalities satisfied by θ .

$$(2.1) \quad \theta(x, st) \leq C_p s^p \theta(x, t) \quad 0 < s < 1$$

$$(2.2) \quad \theta(x, st) \leq s \theta(x, t) \quad s > 1.$$

These two properties are among the ones that are usually required for (constant) Orlicz functions in Hardy Theory, see for instance [12, 3, 13]. They guarantee, in particular, that $L^{\log}(\mathbb{R}^n)$, defined as the set of functions f such that

$$\int_{\mathbb{R}^n} \theta(x, |f(x)|) dx < \infty$$

is a vector space. For $f \in L^{\log}(\mathbb{R}^n)$, we define

$$\|f\|_{L^{\log}} := \inf\{\lambda > 0 ; \int_{\mathbb{R}^n} \theta(x, |f(x)|/\lambda) dx \leq 1\}.$$

It is not a norm, since it is not sub-additive. In place of sub-additivity, there exists a constant C such that, for $f, g \in L^{\log}(\mathbb{R}^n)$,

$$\|f + g\|_{L^{\log}} \leq C(\|f\|_{L^{\log}} + \|g\|_{L^{\log}}).$$

On the other hand, it is homogeneous.

The space $L^{\log}(\mathbb{R}^n)$ is a complete metric space, with the distance given by

$$\text{dist}(f, g) := \inf\{\lambda > 0 ; \int_{\mathbb{R}^n} \theta(x, |f(x) - g(x)|/\lambda) dx \leq \lambda\}$$

(see [20], from which proofs can be adapted, and [13]). Because of (2.1), a sequence f_k tends to 0 in $L^{\log}(\mathbb{R}^n)$ for this distance if and only if $\|f_k\|_{L^{\log}}$ tends to 0.

Before stating our first proposition on products, we need some notations related to the space $BMO(\mathbb{R}^n)$. For Q a cube of \mathbb{R}^n and f a locally integrable function, we note f_Q the mean of f on Q . We recall that a function f is in $BMO(\mathbb{R}^n)$ if

$$\|f\|_{BMO} := \sup_Q \frac{1}{|Q|} \int_Q |f - f_Q| dx < \infty.$$

We note $\mathbb{Q} := [0, 1]^n$ and, for f a function in $BMO(\mathbb{R}^n)$,

$$\|f\|_{BMO^+} := |f_{\mathbb{Q}}| + \|f\|_{BMO}.$$

This is a norm, while the BMO norm is only a norm on equivalent classes modulo constants.

The aim of this section is to prove the following proposition, which replaces Hölder Inequality in our context.

Proposition 2.1. *Let $f \in L^1(\mathbb{R}^n)$ and $g \in BMO(\mathbb{R}^n)$. Then the product fg is in $L^{\log}(\mathbb{R}^n)$. Moreover, there exists some constant C such that*

$$\|fg\|_{L^{\log}} \leq C \|f\|_{L^1} \|g\|_{BMO^+}.$$

Proof. It is easy to adapt the proof given in [2], which leads to a weaker statement. We prefer to give a complete proof here, which has the advantage to be easier to follow than the one given in [2]. We first restrict to functions f of norm 1 and functions g such that $g_{\mathbb{Q}} = 0$ and $\|g\|_{BMO} \leq \alpha$ for some uniform constant α . Let us prove in this case the existence of a uniform constant δ such that

$$(2.3) \quad \int_{\mathbb{R}^n} \theta(x, |f(x)g(x)|) dx \leq \delta.$$

The constant α is chosen so that, by John-Nirenberg inequality, one has the inequality

$$\int_{\mathbb{R}^n} \frac{e^{|g|}}{(e + |x|)^{n+1}} dx \leq \kappa,$$

with κ a uniform constant that depends only of the dimension n (see [21]). Our main tool is the following lemma.

Lemma 2.1. *Let $M \geq 1$. The following inequality holds for $s, t > 0$,*

$$(2.4) \quad \frac{st}{M + \log(e + st)} \leq e^{t-M} + s.$$

Proof. By monotonicity it is sufficient to consider the case when $s = e^{t-M}$. More precisely, it is sufficient to prove that

$$\frac{t}{M + \log(e + te^{t-M})} \leq 1.$$

This is direct when $t \leq M$. Now, for $t \geq M$, the denominator is bounded below by $M + t - M$, that is, by t . \square

Let us go back to the proof of the proposition. We choose $M := (n + 1) \log(e + |x|)$. Then

$$\frac{|f(x)g(x)|}{(n + 1)(\log(e + |x|) + \log(e + |f(x)g(x)|))} \leq \frac{e^{|g(x)|}}{(e + |x|)^{n+1}} + |f(x)|.$$

After integration we get (2.3) with $\delta = (n + 1)(\kappa + 1)$. Let us then assume that $|g_{\mathbb{Q}}| \leq \alpha$ while the other assumptions on f and g are the same. We then write $fg = fg_{\mathbb{Q}} + f(g - g_{\mathbb{Q}})$ and find again the estimate (2.3) with $\delta = (n + 1)(\kappa + 1) + \alpha$. Using (2.1), this means that, for $\|f\|_{L^1} = 1$ and $\|g\|_{BMO^+} = \alpha$ and for $p < 1$, we have the inequality $\|fg\|_{L^{\log}} \leq (\delta C_p)^{1/p}$. The general case follows by homogeneity, with $C = \delta \alpha^{-1}$. \square

Remark that we only used the fact that g is in the exponential class for the weight $(e + |x|)^{-(n+1)}$.

Finally let us define the space $\mathcal{H}^{\log}(\mathbb{R}^n)$. We first define the grand maximal function of a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ as follows. Let \mathcal{F} be the set of functions Φ in $\mathcal{S}(\mathbb{R}^n)$ such that $|\Phi(x)| + |\nabla \Phi(x)| \leq (1 + |x|)^{-(n+1)}$. For $t > 0$, let $\Phi_t(x) := t^{-n} \Phi(\frac{x}{t})$. Then

$$(2.5) \quad \mathcal{M}f(x) := \sup_{\Phi \in \mathcal{F}} \sup_{t > 0} |f * \Phi_t(x)|.$$

By analogy with Hardy-Orlicz spaces, we define the space $\mathcal{H}^{\log}(\mathbb{R}^n)$ as the space of tempered distributions such that $\mathcal{M}f$ in $L^{\log}(\mathbb{R}^n)$. We need the fact that $\mathcal{H}^{\log}(\mathbb{R}^n)$ is a complete metric space. Convergence in $\mathcal{H}^{\log}(\mathbb{R}^n)$ implies convergence in distribution. The space $\mathcal{H}^1(\mathbb{R}^n)$, that is, the space of functions $f \in L^1(\mathbb{R}^n)$ such that $\mathcal{M}f$ in $L^1(\mathbb{R}^n)$, is strictly contained in $\mathcal{H}^{\log}(\mathbb{R}^n)$.

3. PREREQUISITES ON WAVELETS

Let us consider a wavelet basis of \mathbb{R} with compact support. More explicitly, we are first given a $\mathcal{C}^1(\mathbb{R})$ wavelet in Dimension one, called ψ , such that $\{2^{j/2}\psi(2^j x - k)\}_{j,k \in \mathbb{Z}}$ form an $L^2(\mathbb{R})$ basis. We assume that this wavelet basis comes for a multiresolution analysis (MRA) on \mathbb{R} , as defined below (see [17]).

Definition 3.1. A multiresolution analysis (MRA) on \mathbb{R} is defined as an increasing sequence $\{V_j\}_{j \in \mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ with the following four properties

- i) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$ and $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$,
- ii) for every $f \in L^2(\mathbb{R})$ and every $j \in \mathbb{Z}$, $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$,
- iii) for every $f \in L^2(\mathbb{R})$ and every $k \in \mathbb{Z}$, $f(x) \in V_0$ if and only if $f(x-k) \in V_0$,
- iv) there exists a function $\phi \in L^2(\mathbb{R})$, called the scaling function, such that the family $\{\phi_k(x) = \phi(x-k) : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0 .

It is classical that, when given an (MRA) on \mathbb{R} , one can find a wavelet ψ such that $\{2^{j/2}\psi(2^j x - k)\}_{k \in \mathbb{Z}}$ is an orthonormal basis of W_j , the orthogonal complement of V_j in V_{j+1} . Moreover, by Daubechies Theorem (see [6]), it is possible to find a suitable (MRA) so that ϕ and ψ are $\mathcal{C}^1(\mathbb{R})$ and compactly supported, ψ has mean 0 and $\int x\psi(x)dx = 0$, which is known as the moment condition. We could content ourselves, in the following theorems, to have ϕ and ψ decreasing sufficiently rapidly at ∞ , but proofs are simpler with compactly supported wavelets. More precisely we assume that ϕ and ψ are supported in the interval $1/2 + m(-1/2, +1/2)$, which is obtained from $(0, 1)$ by a dilation by m centered at $1/2$.

Going back to \mathbb{R}^n , we recall that a wavelet basis of \mathbb{R}^n is found as follows. We call E the set $E = \{0, 1\}^n \setminus \{(0, \dots, 0)\}$ and, for $\lambda \in E$, state $\psi^\lambda(x) = \phi^{\lambda_1}(x_1) \cdots \phi^{\lambda_n}(x_n)$, with $\phi^{\lambda_j}(x_j) = \phi(x_j)$ for $\lambda_j = 0$ while $\phi^{\lambda_j}(x_j) = \psi(x_j)$ for $\lambda_j = 1$. Then the set $\{2^{nj/2}\psi^\lambda(2^j x - k)\}_{j \in \mathbb{Z}, k \in \mathbb{Z}^n, \lambda \in E}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$. As it is classical, for I a dyadic cube of \mathbb{R}^n , which may be written as the set of x such that $2^j x - k \in (0, 1)^n$, we note

$$\psi_I^\lambda(x) = 2^{nj/2}\psi^\lambda(2^j x - k).$$

We also note $\phi_I = 2^{nj/2}\phi_{(0,1)^n}(2^j x - k)$, with $\phi_{(0,1)^n}$ the scaling function in n variables, given by $\phi_{(0,1)^n}(x) = \phi(x_1) \cdots \phi(x_n)$. In the sequel, the letter I always refers to dyadic cubes. Moreover, we note kI the cube of same center dilated by the coefficient k . Because of the assumption on the supports of ϕ and ψ , the functions ψ_I^λ and ϕ_I are supported in the cube mI .

The wavelet basis $\{\psi_I^\lambda\}$, obtained by letting I vary among dyadic cubes and λ in E , comes from an (MRA) in \mathbb{R}^n , which we still note $\{V_j\}_{j \in \mathbb{Z}}$, obtained by taking tensor products of the one dimensional ones. The functions ϕ_I , taken for a fixed length $|I| = 2^{-jn}$, form a basis of V_j . As in the one dimensional case we note W_j the orthogonal complement of V_j in V_{j+1} . As it is classical, we note P_j the orthogonal projection onto V_j and Q_j the orthogonal projection

onto W_j . In particular,

$$\begin{aligned} f &= \sum_{i \in \mathbb{Z}} Q_i f \\ &= P_j f + \sum_{i \geq j} Q_i f. \end{aligned}$$

4. THE L^2 ESTIMATES FOR THE PRODUCT OF TWO FUNCTIONS

We summarize here the main results of Dobyinsky [8].

Let us consider two L^2 functions f and g , which we express through their wavelet expansions, for instance

$$f = \sum_{\lambda \in E} \sum_I \langle f, \psi_I^\lambda \rangle \psi_I^\lambda.$$

Then, when f and g have a finite wavelet expansion, we have

$$\begin{aligned} (4.1) \quad fg &= \sum_{j \in \mathbb{Z}} (P_j f)(Q_j g) + \sum_{j \in \mathbb{Z}} (Q_j f)(P_j g) + \sum_{j \in \mathbb{Z}} (Q_j f)(Q_j g) \\ &:= \Pi_1(f, g) + \Pi_2(f, g) + \Pi_3(f, g). \end{aligned}$$

The two operators Π_1 and Π_2 are called paraproducts. A posteriori each term of Formula (4.1) can be given a meaning for all functions $f, g \in L^2(\mathbb{R}^n)$. Indeed the two operators Π_1 and Π_2 , which coincide, up to permutation of f and g , extend as bilinear operators from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $\mathcal{H}^1(\mathbb{R}^n)$, see [8], while the operator Π_3 extends to an operator from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$.

The two L^2 estimates are given in the following two lemmas. We sketch their proof for the convenience of the reader as this will be the basis of our proofs in the context of $\mathcal{H}^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$. Details may be found in [8].

Lemma 4.1. *The bilinear operator Π_3 is a bounded operator from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.*

Proof. The series $\sum_{j \in \mathbb{Z}} Q_j f Q_j g$ is normally convergent in $L^1(\mathbb{R}^n)$, with

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \|Q_j f Q_j g\|_{L^1} &\leq \sum_{j \in \mathbb{Z}} \|Q_j f\|_{L^2} \|Q_j g\|_{L^2} \\ &\leq \left(\sum_{j \in \mathbb{Z}} \|Q_j f\|_{L^2}^2 \right)^{1/2} \left(\sum_{j \in \mathbb{Z}} \|Q_j g\|_{L^2}^2 \right)^{1/2} \\ &\leq C \|f\|_{L^2} \|g\|_{L^2}. \end{aligned}$$

This concludes for Π_3 . □

Lemma 4.2. *The bilinear operator Π_1 , a priori well defined for f and g having a finite wavelet expansion, extends to $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into a bounded operator to $\mathcal{H}^1(\mathbb{R}^n)$.*

Proof. Let us recall that one can write

$$P_j f = \sum_{|I|=2^{-jn}} \langle f, \phi_I \rangle \phi_I.$$

This means that $P_j f Q_j g$ can be written as a linear combination of $\psi_I^\lambda \phi_{I'}$, with $|I| = |I'| = 2^{-jn}$. As before, for fixed I , this function is non zero only for a finite number of I' . More precisely, such I' 's can be written as $k2^{-j} + I$, with $k \in K$, where K is the set of points with integer coordinates contained in $(-m, +m]^n$. So $\Pi_1(f, g)$ can be written as a sum in $\lambda \in E$ and $k \in K$ of

$$F_{\lambda, k} := \sum_{j \in \mathbb{Z}} \sum_{|I|=2^{-jn}} \langle f, \phi_{k2^{-j}+I} \rangle \langle g, \psi_I^\lambda \rangle \phi_{k2^{-j}+I} \psi_I^\lambda.$$

At this point, we use the fact that the functions $|I|^{1/2} \phi_{k2^{-j}+I} \psi_I^\lambda$ are of mean zero because of the orthogonality of V_j and W_j . Moreover they are of class $\mathcal{C}^1(\mathbb{R}^n)$ and are obtained from the one for which $I = (0, 1)^n$ through the same process of dilation and translation as the wavelets. So they form what is called a system of molecules. It is well-known (see Meyer's book [17]) that such a linear combination of molecules has its \mathcal{H}^1 norm bounded by C times the \mathcal{H}^1 norm of the linear combination of wavelets with the same coefficients. Namely, we are linked to prove that

$$\left\| \sum_j \sum_{|I|=2^{-jn}} \sum_{\lambda \in E} \langle f, \phi_{k2^{-j}+I} \rangle \langle g, \psi_I^\lambda \rangle 2^{nj/2} \psi_I^\lambda \right\|_{\mathcal{H}^1} \leq C \|f\|_{L^2} \|g\|_{L^2}.$$

We use the characterization of $\mathcal{H}^1(\mathbb{R}^n)$ through wavelets to bound this norm by the L^1 norm of its square function, given by

$$\left(\sum_j \sum_{|I|=2^{-jn}} \sum_{\lambda \in E} |\langle f, \phi_{k2^{-j}+I} \rangle \langle g, \psi_I^\lambda \rangle|^2 2^{nj} |I|^{-1} \chi_I \right)^{1/2}.$$

This function is bounded at x by

$$\sup_{I \ni x} |\langle f, |I|^{-1/2} \phi_I \rangle| \times \left(\sum_j \sum_{|I|=2^{-jn}} \sum_{\lambda \in E} |\langle g, \psi_I^\lambda \rangle|^2 |I|^{-1} \chi_I(x) \right)^{1/2}.$$

The first factor is bounded, up to a constant, by the Hardy Littlewood maximal function of f , which we note Mf . We conclude by using Schwarz inequality, then the maximal theorem to bound the L^2 norm of Mf by the L^2 norm of f , then the fact that the L^2 norm of the second factor is the L^2 norm of g . \square

We will need the expression of $\Pi_1(f, g)$ and $\Pi_2(f, g)$ when f has a finite wavelet expansion while g is only assumed to be in $L^2(\mathbb{R}^n)$. The following lemma is immediate for g with a finite wavelet expansion, then by passing to the limit otherwise.

Lemma 4.3. *Assume that f has a finite wavelet expansion and $Q_j f = 0$ for $j \notin [j_0, j_1]$. For $g \in L^2(\mathbb{R}^n)$, one has*

$$(4.2) \quad \Pi_1(f, g) = \sum_{j=j_0}^{j_1-1} P_j f Q_j g + f \sum_{j \geq j_1} Q_j g$$

$$(4.3) \quad \Pi_2(f, g) = f P_{j_0} g + \sum_{j=j_0}^{j_1-1} Q_j f \left(\sum_{j_0 \leq i \leq j-1} Q_i g \right).$$

5. PRODUCTS OF FUNCTIONS IN $\mathcal{H}^1(\mathbb{R}^n)$ AND $BMO(\mathbb{R}^n)$

Let us first recall the wavelet characterization of $BMO(\mathbb{R}^n)$: if g is in $BMO(\mathbb{R}^n)$, then for all (not necessarily dyadic) cubes R , we have that

$$\left(|R|^{-1} \sum_{\lambda \in E} \sum_{I \subset R} |\langle g, \psi_I^\lambda \rangle|^2 \right)^{1/2} \leq C \|g\|_{BMO},$$

and the supremum over all cubes R of the left hand side is equivalent to the BMO norm of g .

Remark that the wavelet coefficients of a function g in BMO are well defined since g is locally square integrable. The $\langle g, \phi_I \rangle$'s are well defined as well. So $Q_j g$ makes sense, as well as $P_j g$. Indeed, they are sums of the corresponding series in ψ_I^λ or ϕ_I with $|I| = 2^{-jn}$, and at each point only a finite number of terms are non zero.

Moreover, we claim that (4.2) and (4.3) are well defined for f with a finite wavelet expansion and g in $BMO(\mathbb{R}^n)$. This is direct for $\Pi_2(f, g)$. For $\Pi_1(f, g)$, it is sufficient to see that the series $\sum_{j \geq j_1} Q_j g$ converges in $L^2(R)$, where R is a large cube containing the support of f . This comes from the wavelet characterization of $BMO(\mathbb{R}^n)$. Indeed, on R one has

$$\sum_{j_1 \leq j \leq k} Q_j g = \sum_{\lambda \in E} \sum_{I \subset mR, 2^{-nk} \leq |I| \leq 2^{-nj_1}} \langle g, \psi_I^\lambda \rangle \psi_I^\lambda.$$

This is the partial sum of an orthogonal series, that converges in $L^2(\mathbb{R}^n)$.

As a final remark, we find the same expressions for $\Pi_1(f, g)$, $\Pi_2(f, g)$, $\Pi_3(f, g)$ and fg when g is replaced by ηg , where η is a smooth compactly supported function such that η is equal to 1 on a large cube R . Just take R sufficiently large to contain the supports of f , $Q_j f$, and all functions ϕ_I and ψ_I^λ that lead to a non zero contribution in the expressions of the four functions under consideration. Since ηg is in $L^2(\mathbb{R}^n)$, we have the identity (4.1). This leads to the identity

$$(5.1) \quad fg = \Pi_1(f, g) + \Pi_2(f, g) + \Pi_3(f, g).$$

So Theorem 1.1 will be a consequence of the boundedness of the operators $\Pi_1(f, g)$, $\Pi_2(f, g)$ and $\Pi_3(f, g)$.

Before considering this boundedness, we describe the atomic decomposition of the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$, which will play a fundamental role in the proofs.

We recall that a function a is called a (classical) atom of $\mathcal{H}^1(\mathbb{R}^n)$ related to the (not necessarily dyadic) cube R if a is in $L^2(\mathbb{R}^n)$, is supported in R , has mean zero and is such that $\|a\|_{L^2} \leq |R|^{-1/2}$.

For simplicity we will consider atoms that are adapted to the wavelet basis under consideration. More precisely, we call the function a a ψ -atom related to the dyadic cube Q if it is an L^2 function that may be written as

$$(5.2) \quad a = \sum_{I \subset R} \sum_{\lambda \in E} a_{I,\lambda} \psi_I^\lambda$$

such that, moreover, $\|a\|_{L^2} \leq |R|^{-1/2}$. Remark that a is compactly supported in mR and has mean 0, so that it is a classical atom related to mR , up to the multiplicative constant $m^{n/2}$. It is standard that an atom is in $\mathcal{H}^1(\mathbb{R}^n)$ with norm bounded by a uniform constant. The atomic decomposition gives the converse.

Theorem 5.1 (Atomic decomposition). *There exists some constant C such that all functions $f \in \mathcal{H}^1(\mathbb{R}^n)$ can be written as the limit in the distribution sense and in \mathcal{H}^1 of an infinite sum*

$$(5.3) \quad f = \sum_{\ell} \mu_{\ell} a_{\ell}$$

with a_{ℓ} ψ -atoms related to some dyadic cubes R_{ℓ} and μ_{ℓ} constants such that

$$\sum_{\ell} |\mu_{\ell}| \leq C \|f\|_{\mathcal{H}^1}.$$

Moreover, for f with a finite wavelet series, we can choose an atomic decomposition with a finite number of atoms a_{ℓ} , which have also a finite wavelet expansion extracted from the one of f .

This theorem is a small variation of a standard statement. The second part may be obtained easily by taking the atomic decomposition given in [11], Section 6.5. Remark that the interest of dealing with finite atomic decompositions has been underlined recently, for instance in [15, 16].

We want now to give sense to the decomposition (4.1) for $f \in \mathcal{H}^1(\mathbb{R}^n)$ and $g \in BMO(\mathbb{R}^n)$. We will do it when f has a finite wavelet expansion.

Let us first consider that two operators Π_1 and Π_3 .

Theorem 5.2. Π_3 extends into a bounded bilinear operator from $\mathcal{H}^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$.

Proof. We consider f with a finite wavelet expansion and $g \in BMO(\mathbb{R}^n)$, so that $\Pi_3(f, g)$ is well defined as a finite sum in j . Let us give an estimate of its L^1 -norm. We use the atomic decomposition of f given in (5.3), that is,

$f = \sum_{\ell=1}^L \mu_\ell a_\ell$ where each a_ℓ is a ψ -atom related to the dyadic cube R_ℓ and $\sum_{\ell=1}^L |\mu_\ell| \leq C\|f\|_{\mathcal{H}^1}$. Recall that each atom has also a finite wavelet expansion extracted from the one of f . From this, it is sufficient to prove that, for a ψ -atom a , which is supported in R and has L^2 norm bounded by $|R|^{-1/2}$, we have the estimate

$$(5.4) \quad \|\Pi_3(a, g)\|_{L^1} \leq C\|g\|_{BMO}.$$

We claim that $\Pi_3(a, g) = \Pi_3(a, b)$, where $b := \sum_{\lambda \in E} \sum_{I \in 2mR} \langle g, \psi_I^\lambda \rangle \psi_I^\lambda$. Indeed, in the wavelet expansion of g we only have to consider at each scale j the terms ψ_I^λ for which $\psi_I^\lambda \psi_{I'}^{\lambda'}$ is not identically 0 for all I' contained in R such that $|I| = |I'| = 2^{-jn}$. In other words we want $mI \cap mI' \neq \emptyset$, which is only possible for I in $2mR$. Now let us recall the wavelet characterization of $BMO(\mathbb{R}^n)$: for all cubes Q , we have that

$$\left(|Q|^{-1} \sum_{\lambda \in E} \sum_{I \subset Q} |\langle g, \psi_I^\lambda \rangle|^2 \right)^{1/2} \leq C\|g\|_{BMO},$$

and the supremum on all cubes Q of the left hand side is equivalent to the BMO norm of g . It follows that the L^2 norm of b is bounded by $Cm^{n/2}|R|^{1/2}\|g\|_{BMO}$. This allows to conclude for the proof of (5.4), using Lemma 4.1. \square

Next we look at Π_1 .

Theorem 5.3. Π_1 extends into a bounded bilinear operator from $\mathcal{H}^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into $\mathcal{H}^1(\mathbb{R}^n)$.

Proof. Again, we consider $\Pi_1(f, g)$ for f with a finite wavelet expansion, so that it is well defined by (4.2). As in the previous theorem we can consider separately each atom. So, as before, let a be such a ψ -atom. One can estimate $\Pi_1(a, g)$ as in the previous theorem. We again claim that $\Pi_1(f, g) = \Pi_1(f, b)$, where $b := \sum_{\lambda \in E} \sum_{I \in 2mR} \langle g, \psi_I^\lambda \rangle \psi_I^\lambda$. We then use Lemma 4.2 to conclude that

$$(5.5) \quad \|\Pi_1(a, g)\|_{\mathcal{H}^1} \leq C\|g\|_{BMO},$$

which we wanted to prove. \square

We now consider the last term.

Theorem 5.4. Π_2 extends into a bounded bilinear operator from $\mathcal{H}^1(\mathbb{R}^n) \times BMO^+(\mathbb{R}^n)$ into $\mathcal{H}^{\log}(\mathbb{R}^n)$.

Proof. The main point is the following lemma.

Lemma 5.1. *let a be a ψ -atom with a finite wavelet expansion related to the cube R and $g \in BMO$. Then we can write*

$$(5.6) \quad \Pi_2(a, g) = h^{(1)} + \kappa g_R h^{(2)}$$

where $\|h^{(1)}\|_{\mathcal{H}^1} \leq C\|g\|_{BMO}$ and $h^{(2)}$ is an atom related to mR . Here g_R is the mean of g on R and κ a uniform constant, independent of a and g .

Let us conclude from the lemma, which we take for granted for the moment. Let $f = \sum_{\ell=1}^L \mu_\ell a_\ell$ be the atomic decomposition of the function f , which has a finite wavelet expansion. Let us prove the existence of some uniform constant C such that

$$(5.7) \quad \left\| \mathcal{M} \left(\sum_{\ell=1}^L \mu_\ell \Pi_2(a_\ell, g) \right) \right\|_{L^{\log}} \leq C \|g\|_{BMO^+} \left(\sum_{\ell=1}^L |\mu_\ell| \right).$$

With obvious notations, we conclude directly for terms $h_\ell^{(1)}$, using the fact that $L^1(\mathbb{R}^n)$ is contained in $L^{\log}(\mathbb{R}^n)$. So it is sufficient to prove that

$$\left\| \mathcal{M} \left(\sum_{\ell=1}^L \mu_\ell g_{R_\ell} h_\ell^{(2)} \right) \right\|_{L^{\log}} \leq C \|g\|_{BMO^+} \left(\sum_{\ell=1}^L |\mu_\ell| \right).$$

At this point we proceed as in [2]. We use the inequality

$$\mathcal{M} \left(\sum_{\ell=1}^L \mu_\ell g_{R_\ell} h_\ell^{(2)} \right) \leq \sum_{\ell=1}^L |\mu_\ell| g_{R_\ell} \mathcal{M} \left(h_\ell^{(2)} \right).$$

Then we write $g_{R_\ell} = g + (g_{R_\ell} - g)$. For the first term, that is,

$$|g| \left(\sum_{\ell=1}^L |\mu_\ell| \mathcal{M} \left(h_\ell^{(2)} \right) \right),$$

we use the generalized Hölder inequality given in Proposition 2.1. Indeed, g is in $BMO(\mathbb{R}^n)$ and the function $\mathcal{M}(a)$, for a an atom, is uniformly in L^1 , so that $\sum_{\ell=1}^L |\mu_\ell| \mathcal{M} \left(h_\ell^{(2)} \right)$ has norm in L^1 bounded by $C \sum_{\ell=1}^L |\mu_\ell|$. To conclude for (5.7), it is sufficient to prove that

$$\left\| \sum_{\ell=1}^L |\mu_\ell| |g - g_{R_\ell}| \mathcal{M} \left(h_\ell^{(2)} \right) \right\|_{L^1} \leq C \sum_{\ell=1}^L |\mu_\ell|.$$

This is a consequence of the following uniform inequality, valid for $g \in BMO(\mathbb{R}^n)$ and a an atom adapted to the cube R :

$$\int_{\mathbb{R}^n} |g - g_R| \mathcal{M}(a) dx \leq C \|g\|_{BMO}.$$

To prove this inequality, by using invariance through dilation and translation, we may assume that R is the cube \mathbb{Q} . We conclude by using the following classical lemma.

Lemma 5.2. *Let a be a classical atom related to the cube \mathbb{Q} and g be in $BMO(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} |g - g_{\mathbb{Q}}| \mathcal{M}(a) dx \leq C \|g\|_{BMO}.$$

Proof. We cut the integral into two parts. By Schwarz Inequality and the boundedness of the operator \mathcal{M} on $L^2(\mathbb{R}^n)$, we have

$$\begin{aligned} \int_{|x| \leq 2} |g - g_{\mathbb{Q}}| \mathcal{M}(a) dx &\leq C \left(\int_{2\mathbb{Q}} |g - g_{\mathbb{Q}}|^2 dx \right)^{1/2} \|a\|_{L^2} \\ &\leq C \|g\|_{BMO}, \end{aligned}$$

here one used $|g_{2\mathbb{Q}} - g_{\mathbb{Q}}| \leq C \|g\|_{BMO}$. Next, for $|x| > 2$ we have the inequality

$$\mathcal{M}(a)(x) \leq \frac{C}{(1 + |x|)^{n+1}},$$

and the classical inequality (see Stein's book [21])

$$\int_{\mathbb{R}^n} \frac{|g - g_{\mathbb{Q}}|}{(1 + |x|)^{n+1}} dx \leq C \|g\|_{BMO}.$$

We have proven (5.7). \square

It remains to prove Lemma 5.1, which we do now.

Proof of Lemma 5.1. Let a be a ψ -atom which is related to the dyadic cube R . Let j_0 be such that $|R| = 2^{-nj_0}$. We assume that a has a finite wavelet expansion, so that $\Pi_2(a, g)$ is given by (4.2) for some $j_1 > j_0$. As before, we can write $\Pi_2(a, g) = aP_{j_0}g + \Pi_2(a, b)$, where b is defined by $b := \sum_{\lambda \in E} \sum_{I \in 2mR} \langle g, \psi_I^\lambda \rangle \psi_I^\lambda$. It follows again from the characterization of BMO function through wavelets that the L^2 norm of b is bounded by $C \|g\|_{BMO} |R|^{1/2}$. We use the L^2 estimate given by Lemma 4.2 to bound uniformly the \mathcal{H}^1 norm of $\Pi_2(a, b)$. This term goes into $h^{(1)}$.

It remains to consider $aP_{j_0}g$. By definition of $P_{j_0}g$, it can be written as $a \sum_I \langle g, \phi_I \rangle \phi_I$, where the sum in I is extended to all dyadic cubes such that $|I| = 2^{-nj_0}$ and $mI \cap mR \neq \emptyset$. There are at most $(2m)^n$ such terms in this sum, and it is sufficient to prove that each of them can be written as $h_1 + \kappa |g_R| h_2$, with h_2 a classical atom related to mQ and h_1 such that $\|h_1\|_{\mathcal{H}^1} \leq C \|g\|_{BMO}$. Let us first remark that for each of these $(2m)^n$ terms, the function $h := |I|^{1/2} \phi_I a$ is (up to some uniform constant) a classical atom related to mR : indeed, it has mean value 0 because of the orthogonality of ϕ_I and $\psi_{I'}$ when $|I'| \leq |I|$ and the norm estimate follows at once. In order to conclude, it is sufficient to prove that $h_1 = (g_R - |I|^{-1/2} \langle g, \phi_I \rangle) h$ has the required property. We conclude easily by showing that $g_R - |I|^{-1/2} \langle g, \phi_I \rangle$ is bounded by $C \|g\|_{BMO}$. But this difference may be written as $\langle \gamma, g \rangle$, where $\gamma := |R|^{-1} \chi_R - |I|^{-1/2} \phi_I$. The function γ has zero mean, is supported in $2mR$ and has L^2 norm bounded by $2|R|^{-1/2}$. Thus, up to multiplication by some uniform constant, it is a classical atom related to the cube $2mR$. It has an \mathcal{H}^1 norm that is uniformly

bounded and its scalar product with g is bounded by the BMO norm of g , up to a constant, as a consequence of the $\mathcal{H}^1 - BMO$ duality.

This concludes for the proof. \square

We have finished the proof of Theorem 5.4, and also of the one of Theorem 1.1. Just take $S = \Pi_3$. \square

6. DIV-CURL LEMMA

The aim of this section is to prove Theorem 1.2. The methods that we develop are inspired by the papers of Dobyinsky in the case of $L^2(\mathbb{R}^n)$. They are generalized in a forthcoming paper of the last author [14].

Let us first make some remarks. By using the decomposition of each product $F_j G_j$ into $S(F_j, G_j) + T(F_j, G_j)$, we already know that all terms $T(F_j, G_j)$ are in $\mathcal{H}^{\log}(\mathbb{R}^n)$. So we claim that it is sufficient to prove that $\sum_{j=1}^n S(F_j, G_j)$ is also in $\mathcal{H}^{\log}(\mathbb{R}^n)$. We first assume that F is in $\mathcal{H}^1(\mathbb{R}^n, \mathbb{R}^n)$ and G in $BMO(\mathbb{R}^n, \mathbb{R}^n)$. Since F is curl-free, we can assume that F_j is a gradient, or, equivalently, $F_j = R_j f$, where R_j is the j -th Riesz transform and $f = -\sum_{j=1}^n R_j(F_j) \in \mathcal{H}^1(\mathbb{R}^n)$ since $\mathcal{H}^1(\mathbb{R}^n)$ is invariant under Riesz transforms. Next, since G is div-free, we have the identity $\sum_{j=1}^n R_j G_j = 0$. So it is sufficient to prove that $S(R_j f, G_j) + S(f, R_j G_j)$ is in $\mathcal{H}^{\log}(\mathbb{R}^n)$ for each j . So Theorem 1.2 is a corollary of the following proposition.

Proposition 6.1. *Let A be an odd Calderón-Zygmund operator. Then, the bilinear operator $S(Af, g) + S(f, Ag)$ maps continuously $\mathcal{H}^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ into $\mathcal{H}^1(\mathbb{R}^n)$.*

Proof. We make a first reduction, which is done by Dobyinsky in [8]. When considering $S(f, g)$ on $\mathcal{H}^1(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$, we can write it as $S(f, g) = h + S_0(f, g)$ with $h \in \mathcal{H}^1(\mathbb{R}^n)$, where

$$(6.1) \quad S_0(f, g) = \sum_{\lambda \in E} \sum_I \langle f, \psi_I^\lambda \rangle \langle g, \psi_I^\lambda \rangle |\psi_I^\lambda|^2.$$

Indeed, $S(f, g) - S_0(f, g)$ may be written in terms of products $\psi_I^\lambda \psi_{I'}^{\lambda'}$, with $|I| = |I'|$, $(I, \lambda) \neq (I', \lambda')$. These functions are of mean 0 because of the orthogonality of the wavelet basis, have L^2 norm bounded, up to a constant, by $|I|^{-1/2}$, and are supported in mI . So they are C times atoms of $\mathcal{H}^1(\mathbb{R}^n)$. Recall that they are non zero only if $I' = k|I|^{1/n} + I$, with $k \in K$, where K is the set of points with integer coordinates contained in $(-m, +m]^n$. So, to prove that $S(f, g) - S_0(f, g)$ is in $\mathcal{H}^1(\mathbb{R}^n)$ it is sufficient to use the fact that, for fixed λ, λ' and k ,

$$\sum_I |\langle f, \psi_I^\lambda \rangle| |\langle g, \psi_{k|I|^{1/n}+I}^{\lambda'} \rangle| \leq C \|f\|_{\mathcal{H}^1} \|g\|_{BMO}.$$

This is a consequence of the wavelet characterization of f in $\mathcal{H}^1(\mathbb{R}^n)$ and g in $BMO(\mathbb{R}^n)$ and the following lemma, which may be found in [10].

Lemma 6.1. *There exists a uniform constant C , such that, for $(a_I)_{I \in \mathcal{D}}$ and $(b_I)_{I \in \mathcal{D}}$ two sequences that are indexed by the set \mathcal{D} of dyadic cubes, one has the inequality*

$$\sum_{I \in \mathcal{D}} |a_I| |b_I| \leq C \left\| \left(\sum_{I \in \mathcal{D}} |a_I|^2 |I|^{-1} \chi_I \right)^{1/2} \right\|_{L^1} \times \sup_{R \in \mathcal{D}} \left(|R|^{-1} \sum_{I \subset R} |b_I|^2 \right)^{1/2}.$$

Let us come back to the proof of the proposition. From this first step, we conclude that it is sufficient to prove that $B(f, g) := S_0(Af, g) + S_0(f, Ag)$ is in $\mathcal{H}^1(\mathbb{R}^n)$. Using bilinearity as well as the fact that $A^* = -A$, we have

$$B(f, g) := \sum_{\lambda \in E} \sum_{\lambda' \in E} \sum_{I, I'} \langle f, \psi_I^\lambda \rangle \langle g, \psi_{I'}^{\lambda'} \rangle \langle A\psi_I^\lambda, \psi_{I'}^{\lambda'} \rangle (|\psi_{I'}^{\lambda'}|^2 - |\psi_I^\lambda|^2).$$

From this point, the proof is standard. An explicit computation gives that $|\psi_{I'}^{\lambda'}|^2 - |\psi_I^\lambda|^2$ is in $\mathcal{H}^1(\mathbb{R}^n)$, with

$$\| |\psi_{I'}^{\lambda'}|^2 - |\psi_I^\lambda|^2 \|_{\mathcal{H}^1} \leq C \left(\log(2^{-j} + 2^{-j'})^{-1} + \log(|x_I - x_{I'}| + 2^{-j} + 2^{-j'}) \right).$$

Here $|I| = 2^{-jn}$ and $|I'| = 2^{-j'n}$, while x_I and $x_{I'}$ denote the centers of the two cubes. Next we use the well-known estimate of the matrix of a Calderón-Zygmund operator (see [MC, Proposition 1]): there exists some $\delta \in (0, 1]$, such that

$$|\langle A\psi_I^\lambda, \psi_{I'}^{\lambda'} \rangle| \leq Cp_\delta(I, I')$$

with

$$p_\delta(I, I') = 2^{-|j-j'|(\delta+n/2)} \left(\frac{2^{-j} + 2^{-j'}}{2^{-j} + 2^{-j'} + |x_I - x_{I'}|} \right)^{n+\delta}.$$

So, by using the inequality

$$\log \left(\frac{2^{-j} + 2^{-j'} + |x_I - x_{I'}|}{2^{-j} + 2^{-j'}} \right) \leq \frac{2}{\delta} \left(\frac{2^{-j} + 2^{-j'} + |x_I - x_{I'}|}{2^{-j} + 2^{-j'}} \right)^{\delta/2},$$

we obtain

$$\|B(f, g)\|_{\mathcal{H}^1} \leq C \sum_{I, I'} |\langle f, \psi_I^\lambda \rangle| |\langle g, \psi_{I'}^{\lambda'} \rangle| p_{\delta'}(I, I')$$

where $\delta' = \delta/2 > 0$. We conclude by using the fact that the almost diagonal matrix $p_{\delta'}(I, I')$ defines a bounded operator on the space of all sequences

$(a_I)_{I \in \mathcal{D}}$ such that $\left(\sum_I |a_I|^2 |I|^{-1} \chi_I \right)^{1/2} \in L^1(\mathbb{R}^n)$.

This is the end of the proof of Theorem 1.2 for $F \in \mathcal{H}^1(\mathbb{R}^n, \mathbb{R}^n)$ and $G \in BMO(\mathbb{R}^n, \mathbb{R}^n)$ with $\text{curl } F = 0$ and $\text{div } G = 0$. Assume now that $\text{div } F = 0$ and $\text{curl } G = 0$. Similarly as above, we have $\sum_{j=1}^n R_j F_j = 0$ and $G_j = R_j g$

where $g = -\sum_{j=1}^n R_j G_j \in BMO(\mathbb{R}^n)$ since $BMO(\mathbb{R}^n)$ is invariant under Riesz transforms. Hence,

$$F \cdot G = \sum_{j=1}^n (T(F_j, G_j) + S(F_j, G_j)) = \sum_{j=1}^n T(F_j, G_j) + \sum_{j=1}^n (S(F_j, R_j g) + S(R_j F_j, g)).$$

We conclude as before from the proposition. \square

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